University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

PROBLEM SET 3

1.
(based on *Purcell B.1.*)
An electron of rest mass m_e and charge e, moving initially at a constant velocity v, is brought to rest with a uniform deceleration a that lasts for a time $t = v/a$. Compare the electromagnetic energy radiated during the deceleration with the electron's initial kinetic energy. Express this ratio in terms of two lengths: the distance that light travels in time t, and the classical electron radius r_0, defined as

$$r_0 \equiv \frac{e^2}{4\pi\epsilon_0 m_e c^2} \ .$$

To carry out this calculation, you need a formula like Purcell Eq. (B.6) that relates the instantaneous radiated power $P_{\rm rad}$ to the instantaneous acceleration a. In SI units, this formula is

$$P_{\rm rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2a^2}{c^3} \ .$$

(Note that, as far as one has been able to tell experimentally, the electron actually is consistent with having zero radius, and it must have a radius at least several orders of magnitude smaller than the "classical radius" r_0.)

2.
(based on *Purcell B.3.*)
A plane electromagnetic wave with angular frequency ω and electric field amplitude E_0 is incident on an isolated electron. In the resulting sinusoidal oscillation of the electron, the maximum acceleration is $|e|E_0/m$, where e is the electron's charge.

Averaged over many cycles, how much power is radiated by this oscillating charge? (Note that, when the maximum *acceleration* of the electron rather than its maximum *amplitude* is held fixed, the power radiated is independent of the frequency ω.)

Divide this average radiated power by $\epsilon_0 E_0^2 c/2$, the average power density (per unit area of wavefront) in the incident wave. The quotient σ has the dimensions of area and is called a *scattering total cross section.* The energy radiated, or scattered, by the electron, and thus lost from the plane wave, is equivalent to the energy falling on an area σ. (The case considered here, involving a free electron moving nonrelativistically, is often called *Thomson scattering*, after J.J. Thomson, the discoverer of the electron, who first calculated it.)

3.
(based on *Purcell B.4.*)
The master formula

$$P_{\rm rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2a^2}{c^3} \ .$$

is useful for particles moving relativistically, even though $v \ll c$ was assumed in Purcell's derivation of it. To apply it to a relativistic situation, all we have to do is (i) transform to a comoving inertial frame F' in which the particle in question is, at least temporarily, moving slowly; (ii) apply the master formula in that frame; and (iii) transform back to any frame we choose.

Consider a highly relativistic electron ($\gamma \gg 1$) moving perpendicular to a magnetic field $\mathbf{B}$. It is continually accelerated (in a direction perpendicular both to its velocity and to the field), so it must radiate. At what rate does it lose energy? To answer this, transform to a frame F' moving momentarily along with the electron, find E' in that frame, and thereby find $P'_{\rm rad}$.

Now show that, because power is *energy/time*, $P_{\rm rad}$ must be equal to $P'_{\rm rad}$.

This radiation generally is called *synchrotron radiation.* It is both a blessing and a curse. The blessing is that intense beams of UV and X-ray photons are created at synchrotrons designed for that purpose, such as Berkeley Lab's Advanced Light Source. These beams are essential for many studies and uses such as semiconductor lithography. The curse is that synchrotron radiation prevents circular electron accelerators of practical size (up to tens of km in circumference) from exceeding about 10^{11} eV in energy, much weaker than the 10^{12} eV proton beams that have been available at Fermilab for a decade.

4.
Fowles 2.4.
The solution to this problem was sketched in lecture on 7 Sep. In Fowles' notation, $\mathbf{E}_0$ and $\mathbf{B}_0$ are the same as the $\mathbf{E}_1$ and $\mathbf{B}_1$ discussed in class.

5.
Fowles 2.7.

6.
Fowles 2.10.

7.
(a.)
For an ideal linear polarizer with its transmission axis at an arbitrary angle ϕ with respect to the x axis, calculate the Jones matrix. (As usual, the beam direction is z, ϕ is an angle in the xy plane, and ϕ is positive as one rotates from x toward y.)
(b.)
For a linear polarizer, show that its Jones matrix $\mathcal{M}$ is not *unitary*, *i.e.* $\mathcal{M}^*_{ij} \neq \left(\mathcal{M}^{-1}\right)_{ji}$. This means that the action of the wave plate violates *time-reversal invariance*. This makes sense because, for general polarization, the irradiance of a light beam is reduced after passing through the plate.

8.
A *wave plate* is made out of a birefringent crystal whose lattice constants are different in the "fast" and "slow" directions of polarization. This leads to different indices of refraction for the two polarizations. If the x axis is along the "slow" direction of the plate, x polarized light accumulates a phase shift δ with respect to light polarized in the "fast" or y direction, with

$$\delta = \frac{\omega D}{c}(n_x - n_y) \ .$$

Here $n_x > n_y$ if the x direction is "slow", ω is the (fixed) angular frequency of the light, and D is the thickness of the plate. Because the absolute phase of the light is of no experimental interest, the effect of the wave plate is equivalent to multiplying the x component of (complex) $\mathbf{E}_1$ by $\exp{(i\delta/2)}$ and the y component by $\exp{(-i\delta/2)}$.
(a.)
Write the Jones matrix for the ideal wave plate just described.
(b.)
Calculate the Jones matrix for the general case in which the "slow" plate axis lies at an angle ϕ from the x axis, where, as in Problem 7, ϕ is an angle in the xy plane.
(c.)
For the wave plate in (b), show that its Jones matrix $\mathcal{M}$ *is unitary*, *i.e.* $\mathcal{M}^*_{ij} = \left(\mathcal{M}^{-1}\right)_{ji}$. This means that the action of the wave plate *is time-reversal invariant*. For any polarization, the irradiance of a light beam is unaffected by traversing the plate (though its polarization may change dramatically).